Lyapunov spectra of Hamiltonian systems using reduced tangent dynamics

K. Ramasubramanian and M. S. Sriram*
Department of Theoretical Physics, University of Madras, Guindy Campus, Chennai 600025, India
(Received 14 December 1999; revised manuscript received 17 May 2000)

Recently a reduced formulation of tangent dynamics for computing the Lyapunov spectrum of a Hamiltonian system has been developed, using its symplectic structure [Partovi, Phys. Rev. Lett. **82**, 3424 (1999)]. In this paper, we make a detailed numerical comparison of this formulation with a standard algorithm, by applying both to some typical Hamiltonian systems with two and three degrees of freedom. For the latter, we cast the reduced dynamics in a form convenient for computations, and capable of generalization to a higher number of degrees of freedom. The numerical values of the positive Lyapunov exponents coincide in most cases and the convergence rates are also comparable. However, the CPU time required for computation is invariably higher for the reduced formulation, making it less efficient than the standard algorithm.

PACS number(s): 05.45.−a, 02.20.Qs

I. INTRODUCTION

A dynamical system of dimension n has n Lyapunov exponents and n principal directions or eigenvectors, corresponding to a set of nearby trajectories [1]. One of the standard and popular methods to compute the Lyapunov spectrum of a dynamical system involves a Gram-Schmidt reorthonormalizaton (GSR) of the ''tangent vectors'' [2]. Another method, based on the $\mathcal{QR}$ decomposition of the tangent map (where $\mathcal{Q}$ is an orthogonal matrix and $\mathcal{R}$ is an upper triangular matrix) has been studied by several authors [3]. It utilizes representations of orthogonal matrices applied to the tangent map, and does not require the GSR procedure. Though it involves a minimum number of equations, detailed computations of Lyapunov spectra reveal that the $\mathcal{QR}$ method is less efficient than the standard method; however both methods yield essentially the same spectra [4].

Hamiltonian systems are special in the sense that the equations of motion of such systems have a symplectic structure. This results in the remarkable property that the Lyapunov spectrum is invariant with respect to change of sign. Clearly, with symplectic symmetry, a knowledge of the positive Lyapunov exponents is sufficient to determine the full spectrum. A modification of the usual ''$\mathcal{QR}$'' method, incorporating the symplectic structure of Hamiltonian dynamics and christened ''reduced tangent dynamics,'' has been recently proposed by Partovi [5]. For Hamiltonians with one and two degrees of freedom, the method gives reliable, convergent results. However the differential equations corresponding to the reduced tangent dynamics are highly involved and nonlinear and the complexity increases with the number of degrees of freedom. Hence, there is a need to compare the efficiency and accuracy of this method with that of the standard algorithm. This is the aim of the present paper.

The reduced tangent dynamics algorithm would be computationally unviable even for $N=3$ (N being the number of degrees of freedom), without a proper parametrization of $\mathcal{Q}$. In Sec. II, we present the salient features of the algorithm and also an appropriate form for $\mathcal{Q}$ when $N=3$. This can be extended in principle even to cases where $N>3$. In Sec. III, we compare this algorithm with the standard one for computation of Lyapunov spectra, for some typical two- and three-dimensional systems. We conclude the paper with a few remarks. Throughout our paper, we follow the conventions and notations of Ref. [5] as far as possible.

II. REDUCED TANGENT DYNAMICS

Consider a classical system of N degrees of freedom described by the canonical coordinates and momenta $q_i, p_i, i=1,2,\ldots N$, and the Hamiltonian function $H(q_i,p_i,t)$. If $\boldsymbol{\xi}$ stands for the $2N$-dimensional vector $(q,\ p)$, Hamilton's equations can be compactly written in a matrix form as

$$\dot{\boldsymbol{\xi}}=\mathcal{J}\,\boldsymbol{\nabla}_{\boldsymbol{\xi}}H(\boldsymbol{\xi},t). \tag{1}$$

Here $\mathcal{J}$ is a $2N$-dimensional matrix, which is written in terms of $N\times N$ blocks as

$$\mathcal{J}=\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{2}$$

The starting point for the computation of Lyapunov exponents is the tangent map corresponding to Eq. (1):

$$\delta\dot{\boldsymbol{\xi}}=\mathcal{G}\delta\boldsymbol{\xi}, \tag{3}$$

where

$$\mathcal{G}=\mathcal{J}\,\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\nabla}_{\boldsymbol{\xi}}H(\boldsymbol{\xi},t) \tag{4}$$

is the Jacobian matrix. The solution to Eq. (3) can be formally written as

$$\delta\boldsymbol{\xi}(\mathrm{t})=\mathcal{T}[\boldsymbol{\xi}(t),t]\delta\boldsymbol{\xi}(0), \tag{5}$$

where $\mathcal{T}$ is the tangent map whose evolution equation is easily seen to be given by

$$\dot{\mathcal{T}}=\mathcal{G}T. \tag{6}$$

*Email address: tphysmu@imsc.ernet.in

Oseledec's theorem [1] ensures that $\lim_{t\to\infty}\log(\tilde{\mathcal{T}}\mathcal{T})^{1/2t}$ exists; the eigenvalues of this matrix are the $2N$ Lyapunov exponents of the system. Because of the symplectic structure of the Hamilton's equations in Eq. (1), $\mathcal{T}$ is a symplectic matrix satisfying the condition

$$\mathcal{T}\mathcal{J}\tilde{\mathcal{T}}=\mathcal{J}. \tag{7}$$

This would imply that the Lyapunov spectrum is invariant with respect to change of sign [1].

The $\mathcal{QR}$ method applied to Hamiltonian systems (called the ''QSD'' method in Ref. [5]) proceeds as follows. We set $\mathcal{T}=\tilde{\mathcal{Q}}\mathcal{S}$, where $\mathcal{S}$ is symplectic and $\mathcal{Q}$ is both orthogonal and symplectic. $\mathcal{Q}$ can be chosen such that $[\mathcal{Q},\mathcal{J}]=0$ and has the structure

$$\mathcal{Q}=\begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix}, \tag{8}$$

where Q_R and Q_I are $N\times N$ matrices and $Q=Q_R+iQ_I$ is a unitary matrix. $\mathcal{S}$ has the structure

$$\mathcal{S}=\begin{bmatrix} S_1 & P \\ 0 & S_2 \end{bmatrix}, \tag{9}$$

where S_1 is upper triangular, S_2 is lower triangular, and $S_2=\tilde{S}_1^{-1}$. Substituting the decomposition $\mathcal{T}=\tilde{\mathcal{Q}}\mathcal{S}$ in Eq. (6), we find

$$\dot{\mathcal{S}}\mathcal{S}^{-1}-\mathcal{L}=\mathcal{Q}G\tilde{\mathcal{Q}}, \quad \dot{\mathcal{Q}}\tilde{\mathcal{Q}}=\mathcal{L}, \tag{10}$$

where $\mathcal{L}$ is an antisymmetric matrix with the block structure

$$\mathcal{L}=\begin{bmatrix} L_R & L_I \\ -L_I & L_R \end{bmatrix}. \tag{11}$$

Here L_R is antisymmetric, L_I is symmetric, and $L=L_R+iL_I$ is anti-Hermitian.

It can be shown that the Lyapunov exponents λ_i are equal to Λ_i/t in the limit $t\to\infty$, where $\Lambda_i=\ln(\mathcal{S}_{ii})$. Because of the symplectic structure of $\mathcal{S}$, $\Lambda_i=\ln(\mathcal{S}_{ii})$, $\Lambda_{i+N}=-\Lambda_i$, with $i=1,\ldots,N$. The first equation in Eq. (10) can be disassembled into two parts: its diagonal elements describe the time evolution of Λ_i in terms of $(\mathcal{Q}G\tilde{\mathcal{Q}})_{ii}$, whereas the rest provide algebraic expressions for $L_{R,I}$ in terms of $(\mathcal{Q}G\tilde{\mathcal{Q}})_{ij}, i\neq j$ (apart from an inessential differential equation for P). Using the block structure of $\mathcal{Q}$ and $\mathcal{L}$, the equation can be written as $\dot{Q}Q^{\dagger}=L$. Therefore, the reduced tangent dynamics of a Hamiltonian system (that has symplectic symmetry) is given by the equations

$$\dot{\Lambda}_i=(\mathcal{Q}G\tilde{\mathcal{Q}})_{ii}, \quad 1\leqslant i\leqslant N, \quad \dot{Q}Q^{\dagger}=L. \tag{12}$$

These constitute N^2+N equations, with N equations for the Lyapunov exponents and N^2 equations for N^2 angles parametrizing the N-dimensional unitary matrix Q. This should be compared with the $2N^2+N$ equations for a general system when the $\mathcal{QR}$ method is used. In the latter case, there are $2N$ equations for Lyapunov exponents and $2N(2N-1)/2$ equations for the angles parametrizing a $2N$-dimensional orthogonal matrix. In the standard method one has $(2N)^2$ equations for determining the full Lyapunov spectrum. These are the $2N$ copies of $2N$ tangent map equations in Eq. (3). In all the methods, one has to further add the $2N$ equations for the trajectory itself, to what has been listed before.

For Hamiltonians of the form $H(\mathbf{p},\mathbf{q})=\mathbf{p}^2/2+V(\mathbf{q},t)$, the right-hand side in Eq. (12) takes on a simple form. One can show that for $(i,j=1,\ldots,N)$,

$$(\mathcal{Q}G\tilde{\mathcal{Q}})_{ii}=2(Q_ID\tilde{Q}_R)_{ii}, \tag{13}$$

$$(L_R)_{ij}=-2(Q_ID\tilde{Q}_R)_{ij}, \quad i>j,$$

$$(L_I)_{ii}=2(Q_RD\tilde{Q}_R)_{ii}, \tag{14}$$

and

$$(L_I)_{ij}=2(Q_RD\tilde{Q}_R)_{ij}, \quad i\neq j.$$

Here, the N-dimensional matrix D is defined by

$$D=\frac{(I-\nabla_q\nabla_qV)}{2}. \tag{15}$$

Any N-dimensional unitary matrix can be expressed as a product of a phase factor and an SU(N) matrix. For $N=2$, the unitary 2×2 matrix Q can be conveniently parametrized by the exponential form $\exp(i\alpha I)\exp(i\psi\sigma\cdot\hat{\mathbf{n}})$ where $\hat{\mathbf{n}}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\phi)$ is a unit vector and α,ψ,θ, and ϕ are real angles [5]. Then, the reduced tangent dynamics in Eq. (12) reduces to coupled equations for the time evolution of the two positive Lyapunov exponents and the four angles.

For $N=3$, we use the fact that an arbitrary SU(3) matrix U can be written in the following completely disentangled form [6]

$$U=e^{i\sqrt{3}\beta\lambda_8}e^{-i\alpha_3\lambda_3}e^{-i\alpha_2\lambda_2}e^{-i\gamma\lambda_3}e^{-i\nu\lambda_4}e^{i\gamma'\lambda_3}e^{i\alpha_2'\lambda_2}e^{i\alpha_3'\lambda_3}, \tag{16}$$

where λ_i are the following Gellmann matrices that are traceless and Hermitian:

$$\lambda_1=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2=\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

$$\lambda_4=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \lambda_8=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Then Q can be written as

$$Q=e^{-i\alpha I}U. \tag{18}$$

The explicit matrix elements of Q are given by

$$Q_{11}=e^{i(\alpha_3'+\gamma'-\alpha_3-\gamma-\alpha-\beta)}\cos\alpha_2\cos\alpha_2'\cos\nu + e^{i(\alpha_3'-\gamma'-\alpha_3+\gamma-\alpha-\beta)}\sin\alpha_2\sin\alpha_2',$$

$$Q_{12}=e^{i(\gamma'-\alpha_3'-\alpha_3-\gamma-\alpha-\beta)}\cos\alpha_2\sin\alpha_2'\cos\nu - e^{i(\gamma-\alpha_3-\alpha_3'-\gamma'-\alpha-\beta)}\sin\alpha_2\cos\alpha_2',$$

$$Q_{13}=-ie^{-i(\alpha_3+\gamma+\alpha+\beta)}\cos\alpha_2\sin\nu,$$

$$Q_{21}=e^{i(\alpha_3'+\gamma'+\alpha_3-\gamma-\alpha-\beta)}\sin\alpha_2\cos\alpha_2'\cos\nu - e^{i(\alpha_3'-\gamma'+\alpha_3+\gamma-\alpha-\beta)}\cos\alpha_2\sin\alpha_2',$$

$$Q_{22}=e^{i(\alpha_3-\gamma-\alpha_3'+\gamma'-\alpha-\beta)}\sin\alpha_2\sin\alpha_2'\cos\nu + e^{i(\alpha_3+\gamma-\alpha_3'-\gamma'-\alpha-\beta)}\cos\alpha_2\cos\alpha_2',$$

$$Q_{23}=-ie^{-i(\alpha_3-\gamma-\alpha-\beta)}\sin\alpha_2\sin\nu,$$

$$Q_{31}=-ie^{i(\alpha_3'+\gamma'-\alpha+2\beta)}\cos\alpha_2'\sin\nu,$$

$$Q_{32}=-ie^{-i(\alpha_3'-\gamma'+\alpha-2\beta)}\sin\alpha_2'\sin\nu,$$

$$Q_{33}=e^{-i(\alpha-2\beta)}\cos\nu. \tag{19}$$

The matrices Q_R and Q_I, which are the real and imaginary parts of Q, can be found from these expressions. The right-hand sides of the reduced tangent dynamics equations in Eq. (12) can then be found using Eqs. (13), (14), and (19).

Now from Eq. (18), we get

$$\dot{Q}Q^{\dagger}=-i\dot{\alpha}I+\dot{U}U^{\dagger}, \tag{20}$$

since U is unitary. $\dot{U}U^{\dagger}$ is the Maurer-Carton form for the SU(3) group and its explicit form can be derived starting from Eq. (16) and using the transformation properties of λ_i under the SU(3) rotations generated by them. It can be written as

$$\dot{U}U^{\dagger}=i\sum_{i=1}^{8} b_i\lambda_i, \tag{21}$$

where b_i are functions of the angles $\beta,\alpha_3,\alpha_2,\gamma,\nu,\gamma',\alpha_3'$, and α_2' and their derivatives (linear in the latter), $\lambda_1,\lambda_2,\lambda_3,\lambda_4$, and λ_8 are given in Eq. (16) and

$$\lambda_5=\begin{bmatrix}0&0&-i\\0&0&0\\i&0&0\end{bmatrix},\quad \lambda_6=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},$$

$$\lambda_7=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix}. \tag{22}$$

The evolution equations for the nine angles are then derived from the equation $\dot{Q}Q^{\dagger}=L$. Thus we have obtained the coupled equations for the time evolution of the three positive Lyapunov exponents and the nine angles, from the reduced tangent dynamics in Eqs. (12)–(15), and using the representation for Q given by Eqs. (16)–(18).

For $N>3$ also, a completely disentangled form of Q can be used in principle, though the evolution equations for the Lyapunov exponents and the angles become more and more complicated with increasing N.

TABLE I. Comparison of the two methods for some Hamiltonian systems with $N=2$ and 3. The values given in parentheses correspond to the standard method.

System and integration time (T)	Initial condition	Positive Lyapunov spectrum and CPU time (t)
Coupled	$q_1=0.8$	0.1820 (0.1738)
quartic oscr.	$q_2=0.5$	0.0001 (0.0001)
($N=2$)	$p_1=1.0$	
($T=10^5$)	$p_2=1.3$	$t=2410(856)$
Partovi	$q_1=0.0$	
Hamiltonian	$q_2=0.0$	0.311 40 (0.319 80)
$f=0$	$p_1=0.0$	0.000 04 (0.000 00)
($N=2$)	$p_2=\sqrt{2}$	
$T=8\times10^5$	$E=1.00$	$t=22\,895(12\,333)$
Partovi	$q_1=0.0$	
Hamiltonian	$q_2=0.0$	0.1731 (0.1181)
$f=0.5$	$p_1=0.0$	0.0061 (0.0031)
($N=2$)	$p_2=\sqrt{2}$	
$T=10^6$		$t=39\,360(28\,109)$
	$q_1=2.4$	
	$q_2=1.4$	0.6582 (0.6468)
Yang-Mills	$q_3=0.4$	0.2012 (0.1959)
system	$p_1=0.4$	0.0006 (0.0001)
($N=3$)	$p_2=1.4$	
$T=5\times10^4$	$p_3=2.4$	$t=36\,791(3607)$
	$E=10.202$	
	$q_1=1.0$	
	$q_2=0.5$	0.3909 (0.3833)
Yang-Mills	$q_3=0.25$	0.1189 (0.1150)
system	$p_1=1.2$	0.000 18 (0.000 12)
($N=3$)	$p_2=0.6$	
$T=10^5$	$p_3=0.3$	$t=20\,471(1936)$
	$E=1.1091$	

III. COMPARISON OF THE REDUCED TANGENT FORMULATION WITH THE STANDARD ALGORITHM

In this section we compare in detail the Lyapunov spectra computed using the reduced tangent dynamics approach and the standard method. For this purpose, we work with the following Hamiltonian systems with $N=2$ or 3.

(1) Two-dimensional coupled quartic oscillator ($N=2$):

$$H=\frac{p_1^2}{2}+\frac{p_2^2}{2}+q_1^4+q_2^4+\alpha q_1^2q_2^2, \tag{23}$$

where α is a parameter. The system is known to be integrable for $\alpha=0,2$, and 6. We have performed numerical computations for $\alpha=6$ and 8.

(2) Partovi Hamiltonian ($N=2$):

TABLE II. Comparison of convergence of the maximal Lyapunov exponent for the Partovi Hamiltonian with $f=0.0$, corresponding to three initial conditions.

	Initial conditions used for both the methods					
Time	$q_1=0.10$ $q_2=0.60$	$p_1=0.00$ $p_2=0.41$	$q_1=0.20$ $q_2=0.20$	$p_1=1.00$ $p_2=0.25$	$q_1=0.40$ $q_2=1.00$	$p_1=2.00$ $p_2=0.40$
	Standard method	Reduced dynamics	Standard method	Reduced dynamics	Standard method	Reduced dynamics
55 000	0.2286(n)	0.2473(y)	0.2895(y)	0.2852(y)	0.2515(y)	0.2587(y)
60 000	0.2231(n)	0.2460(y)	0.2903(y)	0.2840(y)	0.2524(y)	0.2594(y)
65 000	0.2256(n)	0.2465(y)	0.2911(y)	0.2851(y)	0.2529(y)	0.2586(y)
70 000	0.2112(y)	0.2467(y)	0.2917(y)	0.2846(y)	0.2511(y)	0.2586(y)
75 000	0.2092(y)	0.2480(y)	0.2918(y)	0.2853(y)	0.2522(y)	0.2573(y)
80 000	0.2132(y)	0.2501(y)	0.2923(y)	0.2855(y)	0.2523(y)	0.2558(y)
85 000	0.2122(y)	0.2507(y)	0.2921(y)	0.2868(y)	0.2482(y)	0.2569(y)
90 000	0.2063(y)	0.2479(y)	0.2929(y)	0.2876(y)	0.2458(y)	0.2546(y)
95 000	0.2057(y)	0.2428(y)	0.2933(y)	0.2860(y)	0.2468(y)	0.2538(y)
100 000	0.2076(y)	0.2442(y)	0.2928(y)	0.2866(y)	0.2436(y)	0.2518(y)
CPU time (in s)	1065.28	1748.0	1140.34	1837.17	2058.11	3648.30

$$H=\frac{p_1^2}{2}+\frac{p_2^2}{2}-\frac{(q_1-q_2)^2}{2}+4(q_1^4+q_2^4)+fq_2\cos t, \tag{24}$$

where f represents the strength of the external coupling. We have considered both $f=0$ and $f=0.5$ as in Ref. [5].

(3) Yang-Mills System ($N=3$):

$$H=\frac{p_1^2}{2}+\frac{p_2^2}{2}+\frac{p_3^2}{2}+\frac{(q_1^2q_2^2+q_2^2q_3^2+q_3^2q_1^2)}{2}. \tag{25}$$

This is the Hamiltonian obtained upon reducing the Yang-Mills gauge field theory (of fundamental importance in high-energy physics) [7] to a dynamical system, by considering spatially homogeneous fields and a simple ansatz for the Yang-Mills gauge fields [8]. It has been extensively examined in the literature for studying chaos in gauge theories.

For all the systems, we have used a variable step-size Runge-Kutta routine for integration with an error tolerance $\epsilon\sim10^{-8}$. All the computations were performed on a DEC Alpha based workstation running OpenVMS. The time of integration ranged from $T=50\,000$ to $T=10^6$ depending upon the convergence of the Lyapunov exponents. We also

TABLE III. Comparison of convergence of the maximal Lyapunov exponent for the Partovi Hamiltonian with $f=0.5$, corresponding to three initial conditions.

	Initial conditions used for both the methods					
Time	$q_1=0.10$ $q_2=0.60$	$p_1=0.00$ $p_2=0.41$	$q_1=0.20$ $q_2=0.20$	$p_1=1.00$ $p_2=0.25$	$q_1=0.40$ $q_2=1.00$	$p_1=2.00$ $p_2=0.40$
	Standard method	Reduced dynamics	Standard method	Reduced dynamics	Standard method	Reduced dynamics
165 000	0.2090(y)	0.1725(n)	0.1861(n)	0.2252(y)	0.1462(n)	0.1808(n)
180 000	0.2081(y)	0.1713(n)	0.1922(n)	0.2262(y)	0.1525(n)	0.1770(y)
195 000	0.2056(y)	0.1742(n)	0.1975(y)	0.2251(y)	0.1490(n)	0.1748(y)
210 000	0.2057(y)	0.1789(n)	0.2019(y)	0.2223(y)	0.1507(n)	0.1780(y)
225 000	0.2031(y)	0.1847(y)	0.2026(y)	0.2224(y)	0.1569(n)	0.1760(y)
240 000	0.2023(y)	0.1876(y)	0.2028(y)	0.2225(y)	0.1630(n)	0.1685(y)
255 000	0.2027(y)	0.1887(y)	0.2060(y)	0.2230(y)	0.1686(y)	0.1680(y)
270 000	0.2028(y)	0.1883(y)	0.2063(y)	0.2232(y)	0.1730(y)	0.1702(y)
285 000	0.2032(y)	0.1912(y)	0.2069(y)	0.2242(y)	0.1770(y)	0.1741(y)
300 000	0.1993(y)	0.1912(y)	0.2047(y)	0.2256(y)	0.1764(y)	0.1772(y)
CPU time (in s)	6151.84	11 225.61	5813.22	9896.64	6005.35	12 441.7

TABLE IV. Comparison of convergence of the maximal Lyapunov exponent in the Yang-Mills system corresponding to three initial conditions.

	Initial conditions used for both the methods					
Time	$q_1=0.50$ $q_2=0.40$ $q_3=0.20$	$p_1=0.50$ $p_2=0.40$ $p_3=0.20$	$q_1=0.90$ $q_2=0.60$ $q_3=0.30$	$p_1=0.30$ $p_2=0.60$ $p_3=0.90$	$q_1=1.20$ $q_2=0.34$ $q_3=0.64$	$p_1=1.40$ $p_2=0.34$ $p_3=0.64$
	Standard method	Reduced dynamics	Standard method	Reduced dynamics	Standard method	Reduced dynamics
6000	0.2872(n)	0.2790(n)	0.3601(y)	0.3743(n)	0.4320(y)	0.4406(n)
10 000	0.2798(y)	0.2721(y)	0.3533(y)	0.3681(y)	0.4269(y)	0.4469(n)
14 000	0.2752(y)	0.2715(y)	0.3577(y)	0.3681(y)	0.4336(y)	0.4446(n)
18 000	0.2729(y)	0.2708(y)	0.3607(y)	0.3688(y)	0.4369(y)	0.4437(n)
22 000	0.2755(y)	0.2612(y)	0.3637(y)	0.3636(y)	0.4370(y)	0.4373(y)
26 000	0.2766(y)	0.2653(y)	0.3617(y)	0.3633(y)	0.4379(y)	0.4204(y)
30 000	0.2734(y)	0.2592(y)	0.3545(y)	0.3634(y)	0.4294(y)	0.4236(y)
34 000	0.2730(y)	0.2586(y)	0.3584(y)	0.3653(y)	0.4263(y)	0.4242(y)
38 000	0.2726(y)	0.2604(y)	0.3590(y)	0.3635(y)	0.4199(y)	0.4222(y)
42 000	0.2732(y)	0.2622(y)	0.3608(y)	0.3634(y)	0.4215(y)	0.4185(y)
46 000	0.2716(y)	0.2646(y)	0.3594(y)	0.3404(y)	0.4195(y)	0.4209(y)
50 000	0.2661(y)	0.2659(y)	0.3612(y)	0.3442(y)	0.4168(y)	0.4217(y)
CPU time (in s)	602.33	10 354.09	731.44	40 478.47	1068.23	20 205.23

noted the CPU time taken for each case for both algorithms. This is the actual time taken by the CPU to accomplish a specific process (independent of the other processes running in the system). The details of the comparison of the positive Lyapunov spectra computed using the two methods are summarized in Table I. The two methods yield essentially the same Lyapunov spectra. We will comment on the difference in one case, later.

We make a detailed comparison between the convergence properties of the Lyapunov exponents computed using the two methods by taking up five initial conditions each corresponding to $f=0.0$ and $f=0.5$ for the Partovi system ($N=2$) and 10 initial conditions for the Yang-Mills system ($N=3$). The results for the maximal Lyapunov exponent corresponding to three initial conditions for each of these are displayed in Tables. II–IV. Our criterion of the convergence of the Lyapunov exponent is that the Lyapunov function varies within 5% for a ''sufficiently long interval of time.'' The latter will be clear from the context. This is the reason for the fact that the time of integration depends upon the system and even on the parameters and is indicated in the tables.

The letter $y(n)$ in the parentheses to the right of the maximal Lyapunov exponent indicates that the value of the exponent at that time of integration differs by less (more) than 5% of the ''converged Lyapunov exponent.'' The latter is taken to be the average value of the exponent between $t=40\,000$ and 50 000 for the Yang-Mills system, between $t=80\,000$ and 100 000 for the Partovi system with $f=0.0$, and between $t=240\,000$ and 300 000 for the Partovi system with $f=0.5$. The displayed results are representative and the convergence properties of other exponents and other initial conditions are no different.

In most cases, both the values of the exponents and the convergence rates obtained by the two methods are comparable. The Partovi Hamiltonian has negative terms in it. This leads to a larger phase space and a greater possibility for the propogation of errors. It is perhaps due to this that the maximal Lyapunov exponent obtained by the two methods, considerably differ in one case (refer to the third example in Table. I). With our data it is not possible to say which method is more reliable. However the CPU time is higher by a factor of 1.5 to 2 on an average with the reduced formulation for this system with $N=2$. In Fig. 1, we give the maximal Lyapunov exponent computed using the two methods as a function of time, corresponding to a typical initial condition, and for $f=0.5$.

The convergence of the Lyapunov exponent in the Yang-Mills system ($N=3$) seems to be much faster. The maximal Lyapunov exponent computed using the standard method

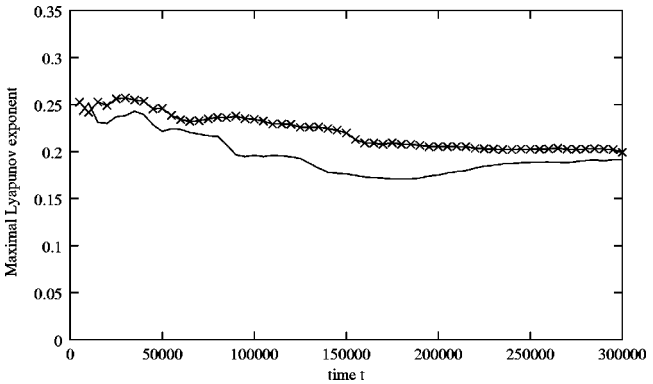

FIG. 1. Plot of the maximal Lyapunov exponent for the Partovi Hamiltonian with $f=0.5$ and the initial conditions $q_1=0.1$, $q_2=0.6$, $p_1=0.0$, and $p_2=0.41$. The lines with and without the ''x'' marks correspond to the standard and the reduced tangent dynamics algorithms, respectively.

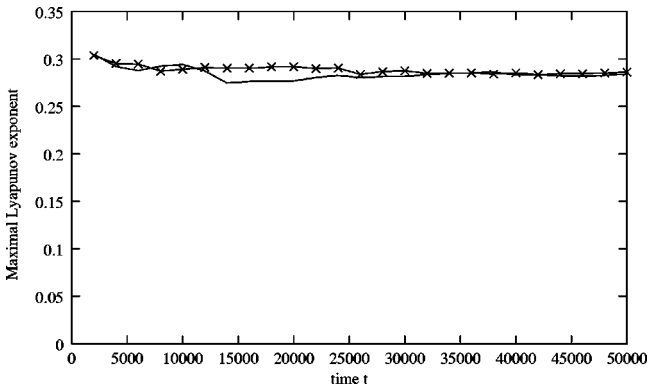

FIG. 2. Plot of the maximal Lyapunov exponent for the Yang-Mills system with $E=0.3192$ and the initial conditions $q_1=0.6$, $q_2=0.4$, $q_1=0.2$, $p_1=0.6$, $p_2=0.4$, and $p_3=0.2$. The lines with and without the ''x'' marks correspond to the standard and the reduced tangent dynamics algorithms, respectively.

converges to within 5% after t=8000 for all the 10 initial conditions we have considered, whereas this is so for 7 cases with the reduced formulation. The values of the exponents computed using the two methods are more or less the same. This is possibly due to the positive definiteness of the Hamiltonian, resulting in a restricted phase space. However, the reduced formulation is far less efficient for the system with $N=3$. The average CPU time corresponding to the reduced formulation is about 15–40 times the CPU time corresponding to the standard method. In Fig. 2, we give the maximal Lyapunov exponent as a function of time computed using the two methods, corresponding to a typical initial condition of the Yang-Mills system.

We have subjected the new algorithm to other tests. For instance, for the Yang-Mills system, the trajectories corresponding to the initial conditions $q_1=q_2=q_3$ and $p_1=p_2=p_3$ are regular and all the Lyapunov exponents are expected to be zero. Similarly the coupled quartic oscillator system is integrable when $\alpha=6$. The computed values of all the Lyapunov exponents using reduced tangent dynamics are indeed zero in these cases.

The reduced tangent dynamics approach does not require renormalization and reorthogonalization as in the standard method. It also uses a lesser number of equations as it incorporates the $\mathcal{Q}R$ method and also the symplectic structure of Hamiltonian dynamics. However, the equations for the tangent flow are highly nonlinear and involved in this formulation. The degree of complexity only increases with N and this is what makes it less efficient.

ACKNOWLEDGMENT

One of the authors (K.R.) thanks the Council for Scientific and Industrial Research, India for financial support.

[1] See for instance A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer-Verlag, New York, 1983).

[2] I. Shimada and T. Nagashima, Prog. Theor. Phys. **61**, 1605 (1979); G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, Meccanica **15**, 9 (1980); A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica D **16**, 285 (1985).

[3] See for instance, K. Geist, U. Parlitz and W. Lauterborn, Prog. Theor. Phys. **83**, 875 (1990), and references therein; G. Rangarajan, S. Habib and R. D. Ryne, Phys. Rev. Lett. **80**, 3747 (1998).

[4] K. Ramasubramanian and M. S. Sriram, Phys. Rev. E **60**, R1126 (1999); Physica D **139**, 72 (2000).

[5] H. Partovi, Phys. Rev. Lett. **82**, 3424 (1999).

[6] T. J. Nelson, J. Math. Phys. **8**, 857 (1967).

[7] See for instance, *Gauge Theories in Particle Physics*, edited by I. J. R. Aitchison and A. J .G. Hey (Adam Hilger, Bristol, 1989).

[8] T. S. Biro, S. G. Matinyan, and B. Muller, *Chaos and Gauge Field Theory* (World Scientific, Singapore, 1994).